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THE MECHANICS OF EROSION BY LIQUID AND SOLID IMPACT

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Abstract—A solution approach for the general problem of erosion by liquid or solid impact is demonstrated for a simplified attack process where impact areas, e.g. craters, of one size only are generated on the target surface. The erosion process is characterized by the specific loss of material (per impact) which is considered to be a random variable, the mathematical expectation of which as a function of time ("erosion curve") is calculated. Based on the statistical nature of the repetitive loading, the individual impacts are classified according to their effectiveness in removing target material. This effectiveness is measured by the probability $P^N_{\text{rep},i}$ to hit, at the Nth impact, as an ith repetition the same neighborhood, e.g. the same crater, on the target surface. The present solution procedure is an extension of that used in Bargmann [*International Journal of Theoretical and Applied Fracture Mechanics*, Vol. 6, pp. 207–215 (1986)] where the probabilities had been calculated, for the early impacts, via so-called erosion-process configurations, the evolution of which had been shown to be Markov. In the present solution procedure, the general solution for the probabilities is given in closed form, for any time t, and the deterministic calculation of the corresponding values of specific material loss, based on impact dynamics, is outlined. It is demonstrated by an illustrative example that the approach is capable of realistically capturing the time-behavior of the erosion curves for both ductile erosion processes and those governed by fatigue.

1. INTRODUCTION

The problem of erosion of a target surface due to liquid impact, solid impact, or cavitation, is a truly interdisciplinary problem as it requires combined efforts in material science, as well as in fluid dynamics and solid mechanics. Excellent pieces of work have been advanced over the last 60 years, mainly in the areas of material science and fluid dynamics, but it appears that the necessary effort in solid mechanics so far had not been made. A typical surface subjected to erosion, i.e. to loss of target material due to continuous bombardment, is presented in Fig. 1.

The problem of erosion, for example by liquid impact, involves a wide variety of material, solid mechanics and fluid dynamics parameters. When fatigue and brittle failure can be excluded, the erosion process is *ductile* and during an incubation period shows complex details of grain boundary delineation and plastic depression of individual grains below the original surface level, finally leading to a general, fairly uniform undulation of the surface and the formation of small, smooth-edged pits. This has been described in detail by Vyas and Preece (1974). Towards the end of the incubation period, the general undulations which may be considered as resulting from the combined pressure pulses of a large number of collapsing bubbles, or from impinging drops, develop into crater-like depressions with large smooth lips.

Material loss is considered to occur from the lips of the craters by ductile rupture (Fig. 2), and again those pressure pulses may be considered responsible for it. A marked time-dependence in the material loss rate is observed. The change of surface topography as erosion proceeds and its feedback to the hydrodynamic loading (e.g. trapped gas and/or liquid at the bottom of deep craters; changes in impact angle) are considered to be major factors. Another factor may be the change in material behavior due to repetitive impact loading, i.e. work hardening may play a role. For example, an average increase in surface hardness by a factor of two has been reported and almost a cubic dependence of erosion resistance on hardness (see Brunton and Rochester, 1979). All these changes are likely to make removal of ductile material far more effective while erosion proceeds through the layers very close to the surface, i.e. at the early stage (after a possible incubation period).

Alternative, but analogous considerations apply to materials prone to *fatigue* and brittle failure.

1.1. *Erosion curves*

Erosion is frequently characterized by so-called "erosion curves" (Fig. 3), which show the so-called "erosion rate" as a function of time. Turbine manufacturers are usually interested in the integral of this curve, i.e. in the area under the erosion curve, as they incur major penalties as soon as the total material loss over the component lifetime exceeds a critical value. Researchers are interested in the character of the erosion curve itself, the discontinuous time-dependence of which, however, they usually smear out from the beginning. They should be interested in the true character of this curve, involving step-size changes in the material loss (specific losses of material) at discrete points in time.

Since Honegger's (1927) experiments it has been known that erosion by liquid impact does not develop at a constant rate. Honegger presented "erosion curves" which exhibited, after an initial incubation period, a phase of increasing erosion rate (acceleration), followed by a phase of decreasing rate (deceleration). He carefully explained the physics of erosion:

> As long as the surface is smooth, it offers no hold for the impinging drops of water and the water flows off on all sides. Therefore, erosion does not occur for some time. However, as soon as any roughness forms, erosion develops rapidly because the water penetrates the unevenness of the surface at a high pressure due to the impact, and acts very violently. Finally, when the erosion has attained a considerable depth, a layer of water adheres to the now completely roughened surface. This water dampens the impact of subsequent drops so that their destructive action is diminished. The specific erosion consequently decreases after a certain depth has been reached.

Honegger's explanation is essentially the same as that put forward in recent publications on the erosion under drop impingement, at least for ductile materials. Even today, opinions differ about which stage in the erosion process is the most important. The quantitative prediction of the erosion curve still remains the principal objective of the research efforts which aim at establishing admissible, not overly conservative (hydrodynamic), loading conditions.

1.2. *From empirical art towards prediction*

The theoretical approaches advanced so far for the quantitative prediction of the erosion-rate time-behavior were essentially based on *ad hoc* assumptions characterizing overall effects. Practically none of them examined or modelled the actual physical occurrences in a more specific way, so they do not lend themselves to further physical refinement.

Heymann (1967) considered the lifetimes of the top surface and subsurface layers as random variables with assumed probability densities. These densities were supposed to reflect all statistical aspects of the erosion problem. The approach is self-consistent; there is, however, no direct way of making a physical refinement.

The fundamental reasoning in Thiruvengadam and Rudy's (1969) theory is, in a sense, a vicious circle. His "efficiency" of erosion, introduced as a premise, is very closely connected with his "intensity" (or rate) of erosion for which he draws his conclusions. In a way, he thus gives as proof the assumption from which he starts.

Springer (1976) assumed from the outset that the erosion rate is constant with time. His model comes down to the assumption that shorter incubation periods correspond to higher subsequent erosion rates. In many situations this may indeed be a valid assumption for the acceleration phase if an incubation period exists.

The mathematical approach recently advanced by Noskievic (1983) definitely takes us outside the discipline of physics. Notwithstanding numerous allusions to concepts of rigid body dynamics, viscoelasticity and damped vibrations, the whole approach is still a curve-fitting exercise in elementary geometry.

Of course, the general problem of erosion is extremely complex. The particular shape of the erosion curve, for example, depends on the material and (geometric) surface conditions of the solid structure, as well as on the type of (thermo- and) hydrodynamic impact loading. It is obvious that complete, analytical solutions to this general problem cannot be

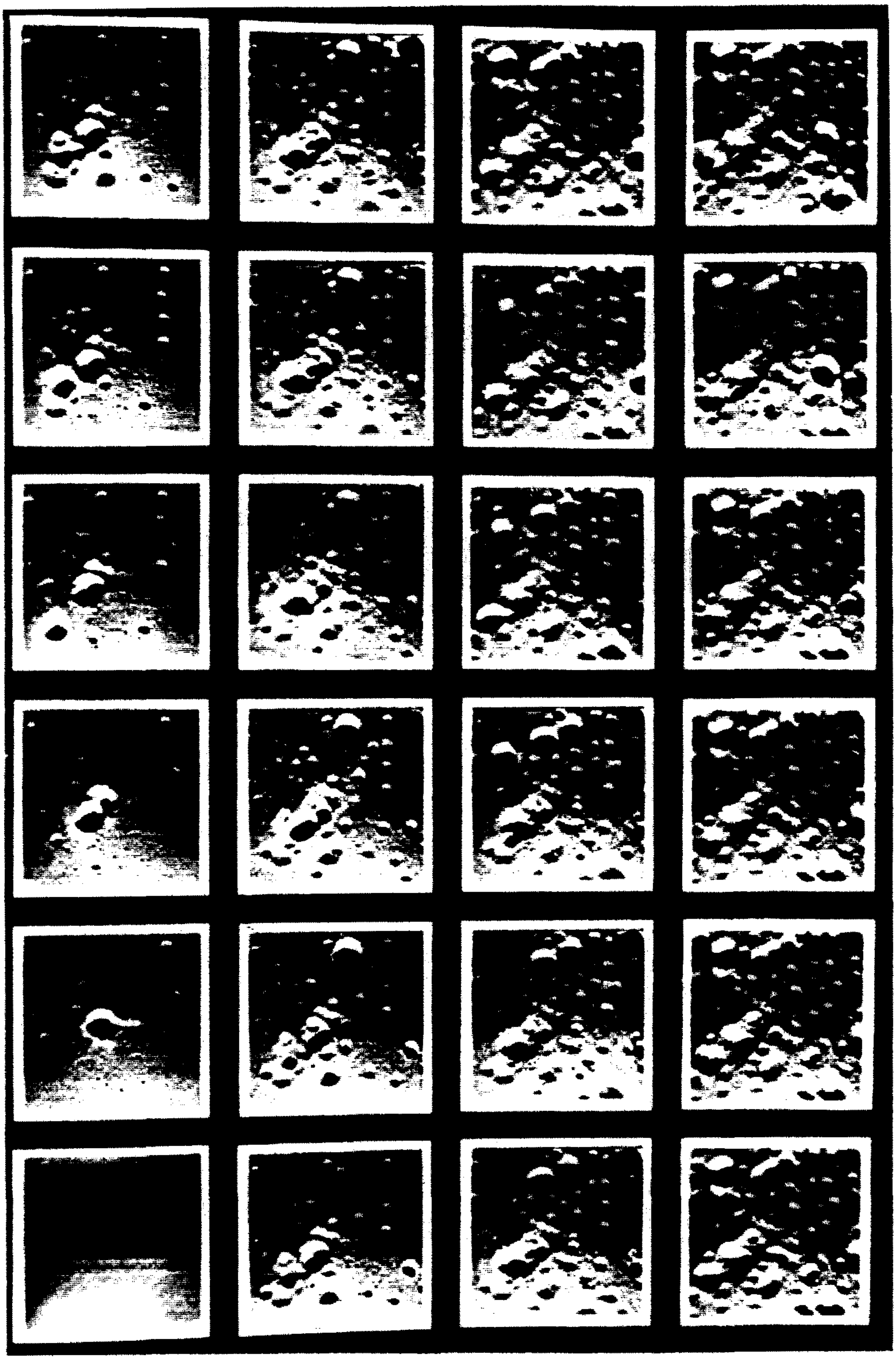

Fig. 1. Erosion: loss of target material due to continuous bombardment. (After Gault, 1970.)

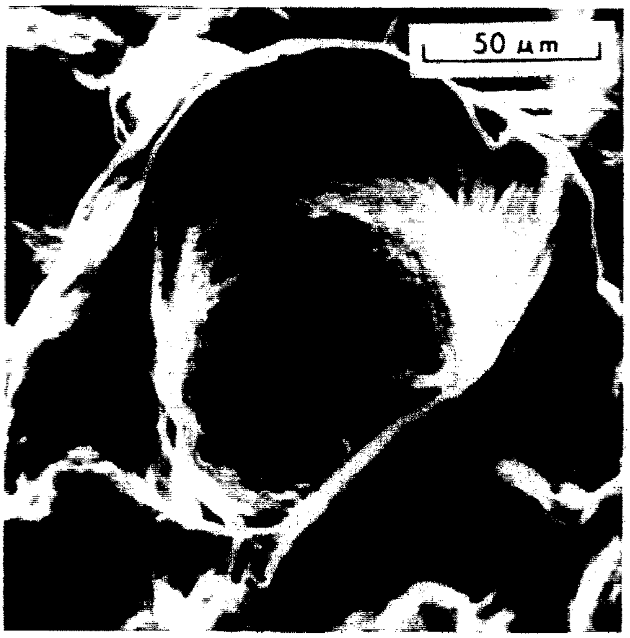

Fig. 2. Crater in ductile target material. The steep rim is particularly prone to failure under a repeated attack. (After Vyas and Preece, 1974.)

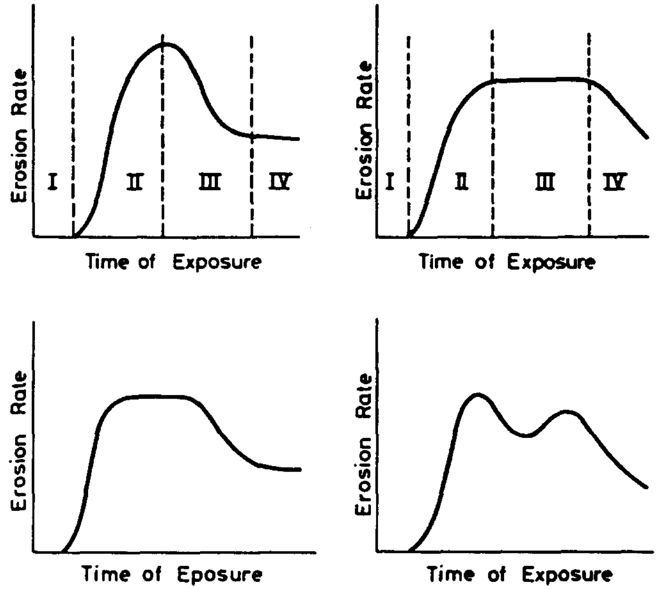

Fig. 3. "Erosion curves": typical erosion rate versus time curves according to different investigators. (After Preece, 1979.)

expected, and an approximate numerical solution is extremely time-consuming and expensive, even with the simplest of assumed material behavior and failure mechanism. Experimental studies also are difficult, time-consuming and expensive.

1.3. *The mechanics of erosion*

It thus remains the hope that, on the basis of preliminary experimental evidence, a highly simplified approach is feasible which still contains the essential features and provides some basic insight and an understanding of the controlling parameters. With a consistent set of concepts it is then possible to guide the experimental work needed; to check, in principle, the results of computer calculations; to advise on the choice of appropriate materials and finally to define the admissible (hydrodynamic) loading conditions.

Thus, the purpose of this paper is twofold. First, we present the general formulation of the problem of erosion in the context of continuum mechanics. This problem is, in principle, a coupled problem of both an attack process and an erosion process, and knowledge of the details of these processes will, in general, be obtained only in a statistical manner. Second, for the particular, practically important situation where both attack process and erosion process may be decoupled, we outline a theory which gives an approximate solution for the erosion process and predicts the mathematical expectation of the erosion rate, i.e. the expected specific erosion, as a function of time ("erosion curve"). This theory is based on the statistical nature of the repetitive attack process of bombarding over an extended target surface, quite similar to raindrops on the roof, and employs a classification of individual impacts according to their effectiveness in removing material. The specific loss of material (per impact) is considered to be a random variable which takes on different values depending on the order of repetition of impacts at the same location on the target. In the case of a ductile erosion process, for example, we assume that the early repetitions of the impact loading are most effective for the removal of material. The solution exhibits the fundamentally stochastic space-time-dependence of erosion. Thus it finally remains, apart from a fluid dynamics aspect of the general problem, a matter of classical solid dynamics, elastoplasticity, and fatigue, to determine, for a given impact, the deterministic, specific loss of target volume.

2. THE GENERAL PROBLEM OF EROSION

The general formulation, within the context of continuum mechanics, of the problem of erosion, has been presented by Bargmann (1988, 1990). It is important to have the general setting of the pluridisciplinary problem in mind, when making the various simplifying approximations in developing the theory and the solution procedures, in order to understand clearly what is going to be neglected and what we would really like to see.

In constructing a general theory of erosion of a solid target, under general conditions of liquid impact, solid impact, or cavitation, we define the *history up to time t of an attack process* by a sequence of fields of surface tractions ("impacts"), $N = 1, 2, 3, \ldots,$

$$\boldsymbol{\sigma}^N = \boldsymbol{\sigma}^N_{(\mathbf{n}')}(\mathbf{r}'_N, t'_N), \quad t'_N \leqslant t \tag{1}$$

defined over a sequence of space × time-neighborhoods on the surface of the target(Fig. 4),

$$U^N \equiv U^N_\rho(\mathbf{r}_N) \times U^N_\tau(t_N) \equiv \{(\mathbf{r}'_N, t'_N) | \mathbf{r}'_N \in U^N_\rho(\mathbf{r}_N) \,\&\, t'_N \in U^N_\tau(t_N)\} \tag{2}$$

where

$$\mathbf{r}_N = \mathbf{f}(N), \quad \rho_N = \phi(N), \quad t_N = g(N), \quad \tau_N = \psi(N) \tag{3}$$

depend on the number N of attacks. These neighborhoods may overlap. We then associate, to each attack process an *erosion process*, defined by both a *material separation*, which is an equation assigning a time-varying material surface $S(t)$ within the original target body, hence the portion $v^{(s)}(t)$ of the target body which, up to time t, has been separated ("been lost"),

$$F^{(s)}(\mathbf{R}, t) = 0, \tag{4}$$

and an *intact motion*, at any time t,

$$\mathbf{r}^{(r)} = \mathbf{r}^{(r)}(\mathbf{R}, t) \quad \text{for} \quad v^{(r)}(t) \equiv v(t) - v^{(s)}(t), \tag{5}$$

which is the motion of that portion $v^{(r)}(t)$ of the original target body $v(t)$ which, at time t, remains still intact, i.e. which has not yet separated (not yet "been lost"). The material surface $S(t)$ just defines that portion, and its history up to time t contains the sequence of spatial neighborhoods $U^N_\rho(\mathbf{r}_N)$.

Since an attack process, for a given structural target configuration and for given parameters of the undisturbed operation (which in the case of liquid impact or cavitation are hydrodynamic parameters like flow velocity, pressure, and air content away from the

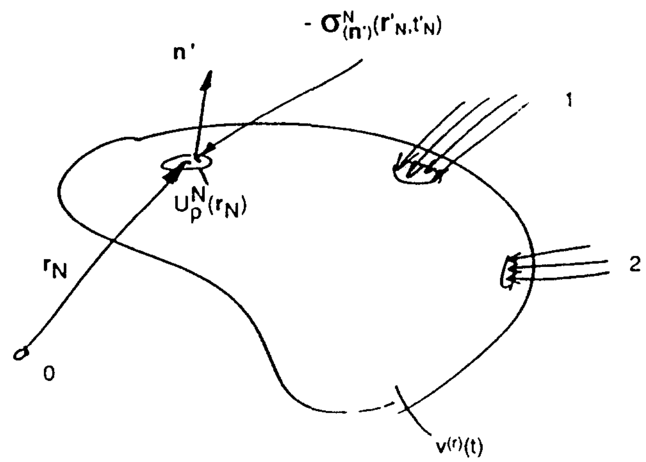

Fig. 4. "Attack process": sequence of surface tractions acting on a sequence of space × time-neighborhoods on the target surface. The associated "erosion process" is defined by the motion of that portion of the target body which, at time t, has not yet been separated.

target surface), is usually determined by very complex, multiphase, thermomechanical and fluid-structure interactions, it cannot be expected that it could be described in a deterministic way, not even for deterministically prescribed target configurations and operating conditions. The attack process and the erosion process are space- and time-dependent stochastic processes (see Bargmann, 1985, 1986).

We may then state the general, *coupled* ***problem of erosion***: *for initially given geometric and material parameters of the solid target, and for given hydrodynamic parameters of the undisturbed operation (or for given mechanical parameters of the undisturbed incoming solid particles), to determine both the stochastic attack process and the stochastic erosion process, at any time t.*

Fortunately, in engineering practice, a complete description of both the attack process and the erosion process is neither necessary nor even desirable. Only certain global features of the erosion process will be of interest, and a number of simplifying assumptions may readily be admitted. Thus one is interested in the total volume $v^{(s)}(t)$ of the target sub-body that has separated up to time t, for a given original target configuration, and for an attack process corresponding to operating conditions.

Frequently, the problem may be simplified by *decoupling* the attack process from the associated erosion process. We may then state the ***first problem of erosion*** as follows: *for initially given geometric and material parameters of the solid target, and for given hydrodynamic parameters of the undisturbed operation (or for given mechanical parameters of the undisturbed incoming solid particles), to determine the stochastic attack process, at any time t.* The ***second problem of erosion*** may then be stated: *for a given stochastic attack process, to determine the stochastic erosion process, at any time t.*

2.1. *Prediction of the erosion curve*

As to the experimental erosion curves, it is important to note that what we really measure are *realizations* of a stochastic process, the specific loss of volume $\mathbf{v}$ being a continuous random variable, depending on time t_N. The most important characteristic parameter is the expected value or mean $\mu(N)$ of this specific loss of volume $\mathbf{v}$, as a function of time $t = t_N$, $N = 1, 2, 3, \ldots,$

$$\mu(N) \equiv E\{\mathbf{v}; N\} = \int_{-\infty}^{\infty} vp(v; N)\,\mathrm{d}v \approx \sum_i v_i p_i(N). \tag{6}$$

From the physical point of view, we need a classification of the individual attacks or impacts according to their effectiveness in removing material. For a numerical evaluation it may be convenient to approximate the continuous random variable $\mathbf{v}$ by a discrete one such that the problem is reduced to determining the weighted sum of discrete values of specific losses of volume v_i (eqn (6), part 3). It will turn out that this sum is much easier determined by interpreting it as a weighted sum of discrete probabilities p_i. We shall thus start with suggesting a certain classification of the individual attacks or impacts, according to their probabilities $p_i(N)$ which may change with the number N of impacts, and only afterwards determine the corresponding values of specific losses of volume v_i.

3. SOLUTION PROCEDURE. PART 1: THE STOCHASTIC SPACE-TIME-DEPENDENCE OF EROSION

We thus consider the specific loss of target material (volume loss per impact) to be a discrete random variable $\mathbf{v}$, taking the values v_i with a certain probability $p_i(N)$. We then need a classification of the individual attacks or impacts according to their effectiveness in removing material, taking into account the statistical nature of the repetitive impact loading. Various types of classification are conceivable. We suggest a particularly simple, approximate classification of the individual attacks or impacts which appears to be supported by experimental evidence. Moreover, the procedure lends itself to straightforward extensions and physical refinement.

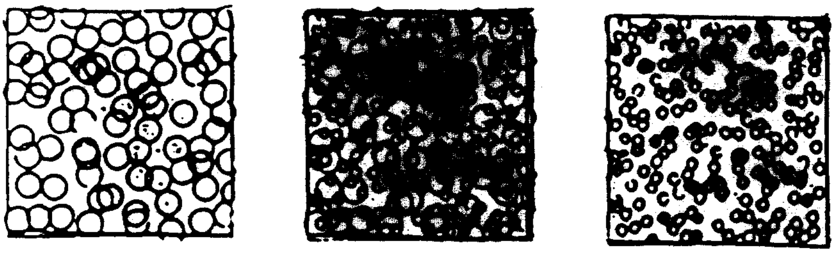

Fig. 5. Target surface under repetitive impacts. The surface at the center corresponds to the superposition of major craters (left) and minor craters (right). In the simplified model, craters of one size only repeatedly cover the surface, and we assume that successive craters are either not at all or totally overlapping.

Let us consider the problem of producing impacts on a solid target (craters say, for a sufficiently ductile material), by random, liquid or solid impingements. The actual situation is extremely complex. Overlapping impacts or craters of various sizes may continuously cover the entire surface (Fig. 5). We adopt a simplified model where (i) craters of *one size* only may repeatedly cover the surface layers of the target; and (ii) successive craters are considered not to overlap at all or to overlap completely, thus *partial overlapping is excluded.*

Let the probability of hitting an existing crater by a next impact be denoted by p. Assume this probability to be constant and equal to the ratio of the area of a crater to the area of the target surface exposed to the attack process, i.e. equal to the constant relative portion $p = A_{\text{impact}}/A_{\text{target}}$ of target surface which is hit at each impact. The smaller the impinging jet or drop or solid particle compared to the size of the crater, the better this assumption is fulfilled. It further contains the idea that wherever the crater is on the surface, the chances of hitting it are the same.

It will turn out that certain, essential features of both the stochastic space-dependence and the stochastic time-dependence of the erosion process are intimately related with each other. Moreover, we shall see that these essential features will be particularly relevant for the establishment of the erosion curve. Thus, having the specific loss of material in mind, it will be not so important *where* exactly on the target surface the next impact occurs, e.g. *where* exactly a new crater is formed, etc., but rather *whether* a new crater is formed, or whether a first repetition of impact at the same spot occurs, etc. (Fig. 6).

Let us thus consider, at the Nth impact, the N separate events: "producing a new crater", "producing a first repetition", "producing a second repetition", ..., "producing a $(N-1)$th repetition". These events together form the certain event. The production of a 0th repetition is defined as the formation of a new crater.

We may then base the classification of the individual impacts on the repetitive nature of impact loading, characterized by the probability of a repeated impact. We may choose the probability $p_i(N)$ to be equal to the probability of producing an ith repetition, at the Nth impact,

$$p_i(N) \equiv P^N_{\text{rep}\,i}, \tag{7}$$

where $P^N_{\text{rep}\,0} \equiv P^N_{\text{new}}$. The problem can thus be reduced to the following fundamental question: *what is the probability to attack ("to hit"), at the Nth impact, as an ith repetition the same neighborhood (e.g. the same crater) on the target surface?*

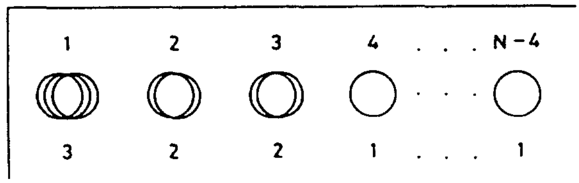

Fig. 6. Erosion-process configuration with one second repetition (3 overlapping craters) and two first repetitions (2 overlapping craters each) after the Nth impact.

Bargmann (1985, 1986) showed that the erosion problem can essentially be reduced to this fundamental question. He introduced the concept of "erosion process configuration" (cf. Fig. 6), and developed a general procedure for the prediction of the erosion process itself. He noted that these configurations have the Markov property, as the next configuration depends only on the current configuration and not on other past configurations. He presented the method for the determination of the different probabilities for a repeated impact $P^N_{rep\,i}$, calculated the values for $N = 1, \ldots, 7$, $i < N$, and showed the general picture of corresponding erosion curves. Indeed, Bargmann's procedure was rigorous and general within the assumptions of his theory, i.e. as to (i) admit one size of craters only, and (ii) exclude partial overlapping. Moreover, the procedure naturally lends itself to any refinement. It was, however, cumbersome to carry out the calculations for impact numbers N higher than 7. Later, Nakkasyan (1985) developed a Monte Carlo program and verified Bargmann's solution. Her program was then used by Lahlou (1988) for the calculation of the solutions up to $N = 100$.

Looking at these results, Bargmann then found the general answer, for arbitrary N, to the fundamental question announced above. Thus we can now present the solution to the problem of the approximate space–time-dependence of the erosion process, for any time t, in analytical, closed form. In fact, a little reflection shows that this probability must be equal to the probability of having hit, from the second to the $(N-1)$st attack, exactly $(i-1)$-times the same portion of area p defined by the (new) crater from the first attack on the target, and to hit, at the Nth attack, again such a stack. This probability is given by

$$P^N_{rep\,i} = \binom{N-1}{i} p^i (1-p)^{N-1-i} \equiv B(i, N-1, p). \tag{8}$$

Equation (8) is easily derived. One way of achieving, at the $(N-1)$st attack, a "stack" of i craters on a particular spot—defined by the (new) crater formed by the first attack—is to have $i-1$ consecutive "successes", each with probability of occurrence p, followed by $N-2-(i-1) \equiv N-1-i$ consecutive "failures". Since each success and failure is independent, the probability of the above sequence is obviously $p^{i-1}(1-p)^{N-1-i}$. Thus, to achieve, at the Nth attack, a stack of $i+1$ craters is given by $p^i(1-p)^{N-1-i}$. Of course, this is only one possible sequence that leads to a stack of i repetitions out of $N-1$ attacks. In general, the number of possible sequences leading to the desired result is equal to the number of combinations of $N-1$ attacks taken i at a time, which is given by $\binom{N-1}{i}$, the different attacks occurring, by definition, in the natural order. All the possible sequences are mutually exclusive, and the derived probability is the probability of the union of these events. Therefore, $B(r; n, p)$ is the sum of the $\binom{N-1}{i}$ identical probabilities $p^i(1-p)^{N-1-i}$ which is of course eqn (8). We note again that (8) is valid for any N, $i < N$, hence for any instant of time t_N, and $p = A_{impact}/A_{target}$ is the probability of "hitting the same area (or crater)", as defined above.

3.1. *Erosion curves*

To illustrate the essential dependence of the probability of a first repetition on the number of impacts, let us choose for the probability of hitting an existing crater the value $p = 0.5$. This rather high value for p allows us to exhibit the typical time-dependence already during the first few impacts. The corresponding values for the probabilities P^N_{new}, $P^N_{rep\,1}$, $P^N_{rep\,2}$, $P^N_{rep\,3}$, etc., are given in Fig. 7. (Note that the smooth curves are valid only at the discrete points $N = 1, 2, 3, \ldots$.)

We immediately observe that a weighted superposition of these probabilities P^N_{new}, $P^N_{rep\,1}$, $P^N_{rep\,2}$, $P^N_{rep\,3}$, etc., results in the typical shape of an erosion rate versus time curve, the number N of impacts being taken as the measure of time (Fig. 7), e.g. the upper curve (032 111 ...).

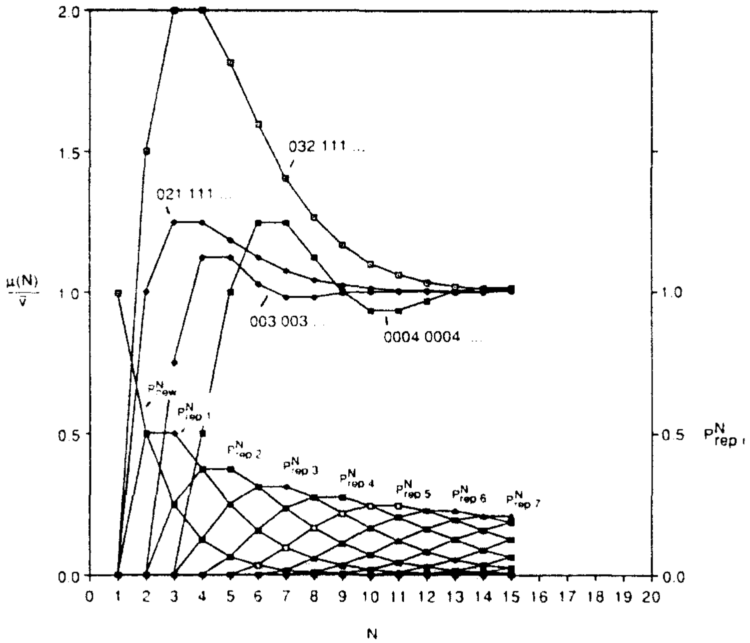

Fig. 7. *Upper half*: Erosion curves. Expected value $\mu(N)$ of specific loss of volume versus number N of impacts ("erosion rate versus time") relative to the standard situation of a constant value $\bar{v}$ being lost at each impact. The four different curves (032 111 . . .), (021 111 . . .), (0004 0004 . . .), (003 003 . . .), correspond to four different influences of early repetitions, hence to four different materials; the first two curves being typical for *ductile* materials, the latter two for materials prone to *fatigue* (cf. Table 1). *Lower half*: Probabilities for producing, at the Nth impact, a new crater, a first, a second, . . . , an ith repetition, $0 < i < N$. For the purpose of illustration, the probability of hitting an existing crater has been assumed to be $p = 0.5$.

Thus all we have to assert is that the specific loss of volume, e.g. v_1 for a first repetition, is much greater than some average specific loss $\bar{v}$ for the higher repetitions; for a very ductile material, this is amply justified throughout the literature. We note that the upper curves in Fig. 7 exhibit all the stages usually found in erosion curves: an incubation period, an acceleration phase, a deceleration phase and a final stationary stage with constant erosion rate.

It is natural to ask how the shape of the erosion curve will be influenced by the higher order repetition probabilities. The values for the probabilities $P^N_{rep\ 2}$, $P^N_{rep\ 3}$, $P^N_{rep\ 4}$, etc., are given in Fig. 7. Again, we observe that weighted superpositions of various $P^N_{rep\ i}$ results in shapes which are typical for erosion rate versus time curves as reported in the literature. Note that the upper curves (032 111 . . .) etc., in Fig. 7, represent the relative expected value (measured in units of a corresponding, stationary, expected value $\bar{v}$) of the specific loss of volume, at the Nth impact hence at time t_N; cf. eqn (6) part 3, where v_i denotes the specific loss of volume for an ith repetition. The curves in Fig. 7 were generated by using the numerical values for the relative effects of the different repetitions as given in Table 1.

Finally, we note that eqn (6) part 3 also allows for the general case where $v_0 \neq 0$, v_0 denoting the specific loss of material when a new crater is produced. It has been observed in certain, less frequent, cases that the erosion curve starts with a maximum value, without an acceleration phase; this could correspond to a significant volume loss even when a new crater is formed (see Heymann, 1967).

4. SOLUTION PROCEDURE, PART II: THE SPECIFIC LOSS OF MATERIAL

We have assumed the specific loss of material v (volume loss per impact) to be a random variable which takes on different values depending on the order of repetition of impacts at the same location on the target. The approximation eqn (6) part 3 of the expected value $\mu(N)$ of the specific loss of material, at any time $t = t_N$, $N = 1, 2, 3, \ldots$, can then be

Table 1. Specific losses of volume v_i, $i = 0, 1, 2, \ldots$, relative to the standard situation of a constant value $\bar{v}$ being lost at each impact

Curve	$\frac{v_0}{\bar{v}}$	$\frac{v_1}{\bar{v}}$	$\frac{v_2}{\bar{v}}$	$\frac{v_3}{\bar{v}}$	$\frac{v_4}{\bar{v}}$	$\frac{v_5}{\bar{v}}$	$\frac{v_6}{\bar{v}}$	$\frac{v_7}{\bar{v}}$
(032 111...)	0	3	2	1	1	1	1	1
(021 111...)	0	2	1	1	1	1	1	1
(0004 0004...)	0	0	0	4	0	0	0	4
(003 003...)	0	0	3	0	0	3	0	0

The four different curves (032 111...), (021 111...), (0004 0004...), (003 003...), correspond to four different influences of early repetitions, hence to four different materials (cf. Fig. 7).

interpreted as a weighted sum of either discrete values of specific losses of volume v_i, or discrete probabilities p_i. The solution exhibits the fundamentally stochastic space–time-dependence of erosion. In solving the problem of erosion, it thus remains a matter of classical mechanics: fluid dynamics, solid dynamics, elastoplasticity, and fatigue, to determine, for a given liquid or solid impact, the deterministic, specific loss of target volume.

4.1. *A typical first problem of erosion*

In solving the first problem of erosion, i.e. in determining the general, stochastic attack process, at any time t, an important step is the deterministic specification of typical surface tractions acting on the target body. Let us illustrate, for the important problem of erosion due to *cavitation*, the type of work to be encountered.

It is well known that, whenever a changing ambient pressure in a liquid falls below a critical value, roughly equal to the vapour pressure, the appearance of cavities occurs. If the pressure in the neighborhood of the effectively vacuous cavity rises above the vapour pressure again, the cavity collapses. When the violent inward motion of the collapsing cavity has come to arrest due to the rising pressure of the gas and vapour in it, a high energy density has been generated in a very small region around the center of the cavity. The energy stored will create an outward-going motion, the cavity starts to rebound. In the early stages of rebound of a spherical cavity (after collapse has come to arrest) an outward-going shock wave may be formed.

We may assess in a simple way the order of magnitude of the shock strength that could be expected, employing Taylor's (1941) self-similar motion type theory for strong shocks, upon the assumption that the total energy carried by the wave remains constant. This should give approximate results at distances from the center where the details of collapse arrest no longer influence the wave shape but where the shock is still strong (i.e. that we may still neglect the pressure ahead of the wave compared to the pressure behind the shock front, $p \gg p_0$). It is a remarkable fact that just for the fluid water the problem admits a closed-form solution. There is a strong attenuation of the peak pressure as the shock wave moves outwards but there are high pressure values at the early stage of rebound confined to a small distance around the center of collapse. For the peak value, at the shock front $r = \xi$, we have, at any time t,

$$p_s/p_0 = 6(R_0/\xi)^3, \tag{9}$$

where R_0 is the initial radius of the cavity and the motion of the shock is given by

$$r = \xi(t) = (150 p_0 R_0^3/\rho_0)^{1/5} t^{2/5}. \tag{10}$$

The result is exact in the limit of a strong shock, $p \gg p_0$, and of an energy deposition taking place instantaneously, at time $t = 0$, at the origin of the sphere. We note that p_0 occurs only in the combination $p_0 R_0^3 \equiv (3/4\pi)E_0$ as the total energy.

Thus, for an outward-going strong shock wave running into water under a pressure p_0 of about an atmosphere ahead of the wave, the maximum pressure at the shock front reaches 6000 atmospheres when the shock front has reached 0.1 of the initial cavity radius R_0 (say $\sim R_m$, the maximum cavity radius, prior to collapse). For $R_0 = 1$ mm, the time of 0.026 μs elapsed from the start of the rebound, the water velocity at the shock front being 380 m s^{-1}, the shock speed 1540 m s^{-1}, and both velocities and peak pressure would be attenuated (the pressure as strong as $1/r^3$) as the shock wave expands further (Fig. 8).

It is well known that reflection of strong shocks results in a considerable increase of the pressure at the wall. As a reminder, we note that for the reflection of a one-dimensional strong shock at a (rigid) wall the pressure ratio is given by

$$p_1/p \sim 3.4 \tag{11}$$

for water, where p_1 is the pressure transmitted to the wall and p corresponds to p_s from (9).

It should be noted that the actual situation in problems of cavitation is of course much more complex: cavities close to a solid wall, in the very final stage of the collapse typically lose their spherical shape and microjets are formed. Brunton and Rochester (1979) have concluded from high speed photographs of single, involuting bubbles that pressures of the order of 5000 atmospheres would be generated when the jet impacts the opposite side of the bubble (for a detached bubble), and nearly twice that value if the bubble is attached and the jet impacts the solid surface directly. Nevertheless, (9) and (11) clearly indicate that

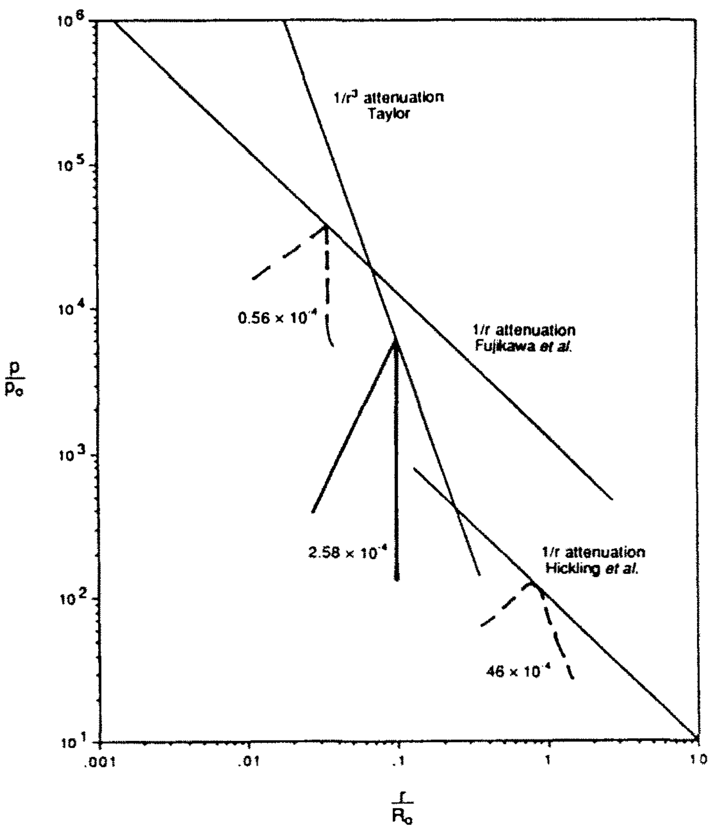

Fig. 8. Variation of pressure with distance from the collapse center at a particular instant in time during the rebound of the cavity. Attenuation of peak pressure with distance and time. The number attached gives the time elapsed from the start of the rebound, expressed in units $R_0(\rho_0/p_0)^{1/2}$ of the order of the collapse time, for a cavity with initial radius $R_0 = 1$ mm. The analytical result of the simple blast wave theory is compared to numerical results of both a theory where evaporation and condensation as well as heat conduction were taken into account (see Fujikawa and Akamatsu, 1980), and a theory based on the assumption that the so-called "kinetic enthalpy" along an outward-going characteristic is inversely proportional to the distance from the center (Kirkwood–Bethe assumption, see Hickling and Plesset, 1964).

the intensities are sufficient to cause micro-craters of dimensions of the order of a grain size in typical metals and corresponding localized material loss.

4.2. *The second problem of erosion*

In solving the second problem of erosion, i.e. in determining, for a given stochastic attack process, the stochastic erosion process, at any time t, we are then faced with the typical matter of classical solid dynamics, elastoplasticity, and fatigue: to determine, for a given impact, the deterministic, specific loss of target volume.

For a *ductile* erosion process, it is reasonable to assume that the early repetitions of the impact loading, e.g. $i = 1, 2, 3, \ldots$, are the most effective ones for the removal of material. Thus, when a crater of volume v_c has been formed, without loss of volume say, we may suppose that a repeated attack will lead to a specific loss of volume v_i which will be some portion f_i of the volume of the crater rim, assumed to be equal to the crater volume, in accordance with the classical assumption on plasticity,

$$v_i \sim f_i v_c, \quad 0 < f_i < 1 \tag{12}$$

for the early repetitions $i = 1, 2, 3, \ldots$.

A number of solutions are proposed which may be used for the assessment of the crater volume v_c, depending on the type of the attack process. For *slow speed* impact, for example, the simple formula

$$v_c/v_p \sim \tfrac{1}{2}\rho_p u_p^2/\bar{Y}, \quad \bar{Y} \sim 3Y \tag{13}$$

may be used, where v_p, ρ_p and u_p denote volume, mass density and velocity of the incoming particle, and Y is the yield strength of the target material (see Johnson, 1972).

A guess for the portion f_i might be a few per cent. A more precise value may either be extracted from experiments or may further be derived on physical grounds, employing particular assumptions on the kinematics and dynamics of the incoming attacks.

For *hypervelocity* impact, an expression for the volume v_c of the penetration at normal incidence that an incoming jet of given volume v_j and mass density ρ_j can achieve in a semi-infinite target of density ρ is easily found by using a hydrodynamic model,

$$v_c/v_j \sim 2\sqrt{(\rho_j/\rho)}. \tag{14}$$

At these high velocities, the pressures created during the impact are assumed to be so high that by comparison static yield strength is altogether negligible, and both jet and target behave like fluids.

For an erosion process governed by *fatigue*, we may assume that some portion a^3 of volume, related to the area of attack of the order of a^2, is lost whenever a typical number of cycles, i.e. impacts on the "same spot", e.g. $i = (1, 2, 3, \ldots) \times 10^n$, typical values for n being 4, 5, or 6, has been reached,

$$v_i \sim a^3, \tag{15}$$

e.g. for $i = 100'000, 200'000, 300'000, \ldots$; and $v_i = 0$ otherwise (see e.g. Johnson 1987; and Hertzberg, 1989, for more details).

5. CONCLUSIONS

The general theory of erosion of a solid target by liquid or solid impact has been presented, employing the concepts of coupled, space- and time-dependent stochastic attack and erosion processes introduced by Bargmann (1988).

A solution approach for the decoupled problem of erosion has been demonstrated for an attack process where impact areas (e.g. craters) of one size only are generated on the target surface. The erosion process is characterized by the specific loss of material (loss of target volume per impact) which is considered to be a random variable, the mathematical expectation of which as a function of time ("erosion curve") is calculated. The statistical nature of the repetitive loading is crucial, as it allows the individual impacts to be classified according to their effectiveness in removing target material. This effectiveness, introduced by Bargmann (1985, 1986), is measured by the probability $P^N_{\text{rep}\,i}$, to hit, at the Nth impact, as an ith repetition the same neighborhood (e.g. the same crater) on the target surface.

In the present solution procedure, the general solution for the probabilities is given in closed form, for any time t, and the deterministic calculation of the corresponding values of specific material loss is outlined. It is demonstrated by an illustrative example that the approach is capable of realistically capturing the time-behavior of the erosion curves for both ductile erosion processes and those governed by fatigue: it exhibits an incubation period, phases of acceleration and deceleration, as well as a final stationary stage of a constant erosion rate.

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REFERENCES

Bargmann, H. W. (1985). Sur l'évolution temporelle de la vitesse d'érosion—une approche probabiliste. Internal Report T-3-85, Laboratoire de mécanique de fluides, EPF Lausanne.

Bargmann, H. W. (1986). On the time-dependence of the erosion rate—a probabilistic approach to erosion. *Int. J. Theor. Appl. Fract. Mech.* **6**, 207–215.

Bargmann, H. W. (1988). On the general theory of erosion. *Z. Angew. Math. Mech.* **68**, T149–T151.

Bargmann, H. W. (1990). Erosion by liquid or solid impact—a random process. In *Book of Lectures* (Tagung Kontinuumsmechanik der festen Körper, 25–31 March 1990). Mathematisches Forschungsinstitut Oberwolfach, BRD.

Brunton, J. H. (1970). In *Proc. 3rd Int. Conf. Rain Erosion* (Edited by A. A. Fyall), p. 821. Royal Aircraft Establishment, Farnborough, U.K.

Brunton, J. H. and Rochester, M. C. (1979). Erosion of solid surfaces by the impact of liquid drops. In *Erosion, Treatise on Materials Science and Technology*, Vol. 16 (Edited by C. M. Preece), pp. 185–248. Academic Press, New York.

Fujikawa, S. and Akamatsu, T. (1980). Effects of the non-equilibrium condensation of vapour on the pressure wave produced by the collapse of a bubble in a liquid. *J. Fluid. Mech.* **97**, 481–512.

Gault, D. E. (1970). Saturation and equilibrium conditions for impact cratering on the lunar surface: criteria and implications. *Radio Sci.* **5**, 273–291.

Hertzberg, R. W. (1989). *Deformation and Fracture Mechanics of Engineering Materials*. Wiley, New York.

Heymann, J. F. (1967). On the time dependence of the rate of erosion due to impingement or cavitation. In *Erosion by Cavitation or Impingement*, pp. 77–110. ASTM STP 408. American Society for Testing and Materials, Philadelphia.

Hickling, R. and Plesset, M. S. (1964). Collapse and rebound of a spherical bubble in water. *The Physics of Fluids* **7**, 7–14.

Honegger, E. (1927). Essais d'érosion des ailettes de turbines à vapeur. *Revue BBC* **14**, 95–104. BBC, Baden.

Johnson, W. (1972). *Impact Strength of Materials*. Edward Arnold, London.

Johnson, K. L. (1987). *Contact Mechanics*. Cambridge University Press, Cambridge.

Lahlou, K. (1988). Sur la classification des sollicitations extérieures en thermomécanique. Projet de 4eme année, Laboratoire de mécanique appliquée, EPF Lausanne.

Nakkasyan, A. (1985). Private Communication. Laboratoire de mécanique de fluides. EPF Lausanne.

Noskievic, J. (1983). The extended mathematical model of cavitation and erosion wear. In *Proc. 6th Int. Conf. on Erosion by Liquid and Solid Impact* (Edited by J. E. Field and N. S. Corney), pp. 701–709. Cambridge, U.K.

Preece, C. M. (1979). Cavitation erosion. In *Erosion, Treatise on Materials Science and Technology*, Vol. 16 (Edited by C. M. Preece), pp. 249–308. Academic Press, New York.

Springer, G. (1976). *Erosion by Liquid Impact*. Wiley, New York.

Taylor, G. I. (1941). The formation of a blast wave by a very intense explosion. Ministry of Home Security, R.C. 210, II-5-153, London.

Thiruvengadam, A. and Rudy, S. L. (1969). Experimental and analytical investigations on multiple liquid impact erosion. NASA CR-1288, Washington.

Vyas, B. and Preece, C. M. (1974). Cavitation-induced deformation of aluminium. In *Erosion, Wear, and Interfaces with Corrosion*, pp. 77–105. ASTM STP 567. American Society for Testing and Materials, Philadelphia.